CORE 4 JUNE 2010

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1.

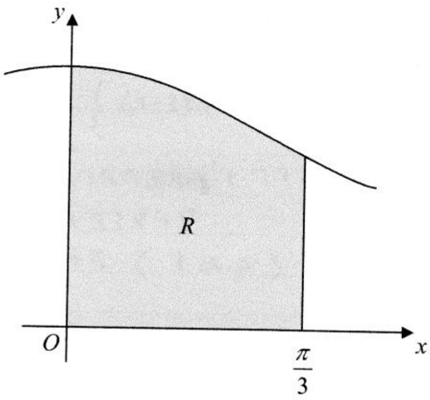

Figure 1

Figure 1 shows part of the curve with equation $y = \sqrt{(0.75 + \cos^2 x)}$. The finite region R, shown shaded in Figure 1, is bounded by the curve, the y-axis, the x-axis and the line with equation $x = \frac{\pi}{3}$.

(a) Complete the table with values of y corresponding to $x = \frac{\pi}{6}$ and $x = \frac{\pi}{4}$.

x	0	$\frac{\pi}{12}$	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$
y	1.3229	1.2973	1.2247	1.1180	1

(2)

(b) Use the trapezium rule

(i) with the values of y at $x = 0$, $x = \frac{\pi}{6}$ and $x = \frac{\pi}{3}$ to find an estimate of the area of R. Give your answer to 3 decimal places.

(ii) with the values of y at $x = 0$, $x = \frac{\pi}{12}$, $x = \frac{\pi}{6}$, $x = \frac{\pi}{4}$ and $x = \frac{\pi}{3}$ to find a further estimate of the area of R. Give your answer to 3 decimal places.

(6)

H 3 5 3 8 6 A 0 2 3 2

Question 1 continued

b) i) $R \approx \frac{1}{2}h\{(y_0+y_2)+2y_1\}$

$= \frac{1}{2} \times \frac{\pi}{6}\{(1.3229+1)+2\times 1.2247\}$

$= 0.2617993 \times 4.7723$

$= 1.2493848$

$= 1.249$ (3 d.p.)

ii) $R \approx \frac{1}{2}h\{(y_0+y_4)+2(y_1+y_2+y_3)\}$

$= \frac{1}{2} \times \frac{\pi}{12}\{(1.3229+1)+2(1.2973+1.2247+1.1180)\}$

$= 0.1308996 \times 9.6029$

$= 1.2570158$

$= 1.257$ (3 d.p)

2. Using the substitution $u = \cos x + 1$, or otherwise, show that

$$\int_0^{\frac{\pi}{2}} e^{\cos x + 1} \sin x \, dx = e(e-1)$$

(6)

$$I = \int_0^{\pi/2} e^{\cos x + 1} \sin x \, dx$$

$u = \cos x + 1$ $\qquad$ $x = 0 \quad u = 2$

$\frac{du}{dx} = -\sin x$ $\qquad$ $x = \pi/2 \quad u = 1$

Replace dx by $-\frac{du}{\sin x}$

$$\therefore I = -\int_2^1 e^u \frac{\sin x \, du}{\sin x}$$

$$= -\int_2^1 e^u du$$

$$= -[e^u]_2^1 = -e + e^2 = e(e-1)$$

H 3 5 3 8 6 A 0 6 3 2

3. A curve C has equation

$$2^x + y^2 = 2xy$$

Find the exact value of $\frac{dy}{dx}$ at the point on C with coordinates $(3, 2)$.

(7)

Differentiate both sides with respect to x

$$\frac{d}{dx}\left(2^x + y^2\right) = \frac{d}{dx}(2xy)$$

$$\therefore (\ln 2) \times 2^x + 2y\frac{dy}{dx} = 2y + 2x\frac{dy}{dx}$$

$$\therefore \frac{dy}{dx} = \frac{(\ln 2) \times 2^x - 2y}{2x - 2y}$$

if $x = 3,\ y = 2,\quad \frac{dy}{dx} = \frac{\ln 2 \times 8 - 4}{6 - 4} = \frac{8\ln 2 - 4}{2}$

$$= 4\ln 2 - 2$$

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4. A curve C has parametric equations

$$x = \sin^2 t, \quad y = 2\tan t, \quad 0 \leqslant t < \frac{\pi}{2}$$

(a) Find $\dfrac{dy}{dx}$ in terms of t.

(4)

The tangent to C at the point where $t = \frac{\pi}{3}$ cuts the x-axis at the point P.

(b) Find the x-coordinate of P.

(6)

$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$

$y = 2\tan t$ $\quad$ $x = \sin^2 t$

$\therefore \frac{dy}{dt} = 2\sec^2 t$ $\quad$ $\frac{dx}{dt} = 2\sin t\cos t$

$\therefore \frac{dy}{dx} = \frac{2\sec^2 t}{2\sin t\cos t} = \frac{\sec^2 t}{\sin t\cos t} = \csc t\sec^3 t$

b) when $t = \frac{\pi}{3}$, $x = \frac{3}{4}$, $y = 2\sqrt{3}$

and $\frac{dy}{dx} = 2 \times \frac{1}{\sqrt{3}} \times 8 = \frac{16}{\sqrt{3}}$

$\therefore$ equation is $(y - 2\sqrt{3}) = \frac{16}{\sqrt{3}}\left(x - \frac{3}{4}\right)$

cuts the x axis where $y = 0$ and $x = x_1$

$\therefore -2\sqrt{3} = \frac{16}{\sqrt{3}}\left(x_1 - \frac{3}{4}\right)$

$\therefore x_1 = -\frac{3}{8} + \frac{3}{4} = \frac{3}{8}$

H 3 5 3 8 6 A 0 1 2 3 2

5. $$\frac{2x^2+5x-10}{(x-1)(x+2)} \equiv A + \frac{B}{x-1} + \frac{C}{x+2}$$

(a) Find the values of the constants A, B and C.

(4)

(b) Hence, or otherwise, expand $\frac{2x^2+5x-10}{(x-1)(x+2)}$ in ascending powers of x, as far as the term in x^2. Give each coefficient as a simplified fraction.

(7)

a) $\frac{2x^2+5x-10}{(x-1)(x+2)} = A + \frac{B}{x-1} + \frac{C}{x+2}$

$\therefore \; 2x^2+5x-10 = A(x-1)(x+2) + B(x+2) + C(x-1)$

$x=1 \Rightarrow 2+5-10 = 3B \Rightarrow B=-1$

$x=-2 \Rightarrow 8-10-10 = -3C \Rightarrow C=4$

$x=0 \Rightarrow -10 = A(-1)(2)+2B-C$

$\Rightarrow -10 = -2A-2-4$

$\Rightarrow 2A = 4$

$\Rightarrow A=2$

b) $\frac{2x^2+5x-10}{(x-1)(x+2)} = 2 - \frac{1}{x-1} + \frac{4}{x+2}$

$= 2 + (1-x)^{-1} + 4(2+x)^{-1}$

$= 2 + (1-x)^{-1} + 2(1+\frac{1}{2}x)^{-1}$

$= 2 + (1+x+x^2+\ldots) + 2(1-\frac{1}{2}x+\frac{x^2}{4}+\ldots)$

$= 2+1+x+x^2+2-x+\frac{x^2}{2}+\ldots$

$= 5+\frac{3x^2}{2}$

6.

$$f(\theta) = 4\cos^2\theta - 3\sin^2\theta$$

(a) Show that $f(\theta) = \frac{1}{2} + \frac{7}{2}\cos 2\theta$.

(3)

(b) Hence, using calculus, find the exact value of $\int_0^{\frac{\pi}{2}} \theta f(\theta)\, d\theta$.

(7)

a) $f(\theta) = 4\cos^2\theta - 3\sin^2\theta$

$= 4\cos^2\theta - 3(1-\cos^2\theta)$

$= 7\cos^2\theta - 3$

$= \frac{7}{2}(2\cos^2\theta - 1) + \frac{7}{2} - 3$

$= \frac{7}{2}\cos 2\theta + \frac{7}{2} - 3$ since $\cos 2\theta = 2\cos^2\theta - 1$

$= \frac{7}{2}\cos 2\theta + \frac{1}{2}$

b) $I = \int_0^{\pi/2} \theta(4\cos^2\theta - 3\sin^2\theta)\, d\theta$

$= \int_0^{\pi/2} \frac{7}{2}\theta\cos 2\theta + \frac{1}{2}\theta \; d\theta$

$= \frac{7}{2}\int_0^{\pi/2} \theta\cos 2\theta\, d\theta + \frac{1}{2}\int_0^{\pi} \theta\, d\theta$

$u = \theta$ $\quad \frac{dv}{d\theta} = \cos 2\theta$

$\frac{du}{d\theta} = 1$ $\quad v = \frac{1}{2}\sin 2\theta$

$\therefore I = \frac{7}{2}\left[\frac{\theta}{2}\sin 2\theta\right]_0^{\pi/2} - \frac{7}{2}\int_0^{\pi/2} \frac{1}{2}\sin 2\theta\, d\theta + \frac{1}{2}\int_0^{\frac{\pi}{2}} \theta\, d\theta$

$= \frac{7}{2}\left[\frac{\theta}{2}\sin 2\theta\right]_0^{\pi/2} + \frac{7}{2}\left[\frac{1}{4}\cos 2\theta\right]_0^{\pi/2} + \frac{1}{2}\left[\frac{\theta^2}{2}\right]_0^{\pi/2}$

$= \frac{7}{2}(0-0) + \frac{7}{2}\left(-\frac{1}{4} - \frac{1}{4}\right) + \left(\frac{\pi^2}{16} - 0\right)$

$= \frac{\pi^2}{16} - \frac{7}{4}$

7. The line l_1 has equation $\mathbf{r} = \begin{pmatrix} 2 \\ 3 \\ -4 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$, where λ is a scalar parameter.

The line l_2 has equation $\mathbf{r} = \begin{pmatrix} 0 \\ 9 \\ -3 \end{pmatrix} + \mu \begin{pmatrix} 5 \\ 0 \\ 2 \end{pmatrix}$, where μ is a scalar parameter.

Given that l_1 and l_2 meet at the point C, find

(a) the coordinates of C.

(3)

The point A is the point on l_1 where $\lambda = 0$ and the point B is the point on l_2 where $\mu = -1$.

(b) Find the size of the angle ACB. Give your answer in degrees to 2 decimal places.

(4)

(c) Hence, or otherwise, find the area of the triangle ABC.

(5)

a) $\underline{r}_1 = \begin{pmatrix} 2 \\ 3 \\ -4 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$

$\underline{r}_2 = \begin{pmatrix} 0 \\ 9 \\ -3 \end{pmatrix} + \mu \begin{pmatrix} 5 \\ 0 \\ 2 \end{pmatrix}$

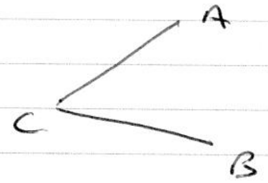
$\underline{r}_1$ and $\underline{r}_2$ meet so that
$2 + \lambda = 5\mu$ — (1)
$3 + 2\lambda = 9$ — (2)
$4 + \lambda = -3 + 2\mu$ — (3)

(2) $\Rightarrow \lambda = 3$

$\therefore$ Meet at $\underline{r} = \begin{pmatrix} 5 \\ 9 \\ -1 \end{pmatrix}$ i.e. at $C(5, 9, -1)$

Question 7 continued

b) $A(2, 3, -4)$
$B(-5, 9, -5)$



$\underset{\sim}{AC} \cdot \underset{\sim}{BC} = AC \times CB \times \cos A\hat{C}B$

$\underset{\sim}{AC} = \begin{pmatrix} 3 \\ 6 \\ 3 \end{pmatrix}$ $\underset{\sim}{BC} = \begin{pmatrix} 10 \\ 0 \\ 4 \end{pmatrix}$

better to use
$\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 5 \\ 0 \\ 2 \end{pmatrix} = 7$
$\sqrt{6}$ $\sqrt{29}$
$\cos A\hat{C}B = \frac{7}{\sqrt{6}\sqrt{29}}$

$\therefore \underset{\sim}{AC} \cdot \underset{\sim}{BC} = 30 + 12 = 42$

$AC = \sqrt{54}$ $BC = \sqrt{116}$

$\therefore \cos A\hat{C}B = \frac{42}{\sqrt{54}\sqrt{116}}$

$\therefore$ $A\hat{C}B = 57.95°$ (2 d.p.)

c) $\text{Area} = \frac{1}{2} \times AC \times BC \times \sin 57.95°$

$= \frac{1}{2} \times \sqrt{54}\sqrt{116} \times 0.8475793$

$= 33.54$

need "my" part (b) method here

8.

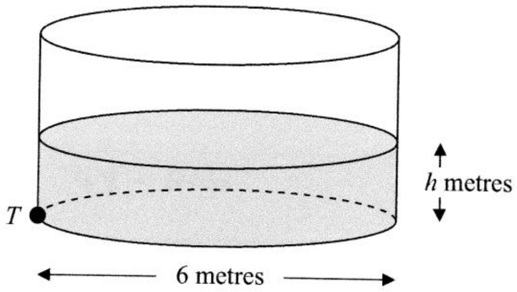


Figure 2

Figure 2 shows a cylindrical water tank. The diameter of a circular cross-section of the tank is 6 m. Water is flowing into the tank at a constant rate of 0.48π m^3 min^{-1}. At time t minutes, the depth of the water in the tank is h metres. There is a tap at a point T at the bottom of the tank. When the tap is open, water leaves the tank at a rate of $0.6\pi h$ m^3 min^{-1}.

(a) Show that t minutes after the tap has been opened

$$75\frac{dh}{dt} = (4-5h)$$

(5)

When $t = 0$, $h = 0.2$

(b) Find the value of t when $h = 0.5$

(6)

a) $\frac{dV}{dt} = 0.48\pi - 0.6\pi h$

but $V = 9\pi h$

$\therefore \frac{dV}{dt} = \frac{d}{dh}(9\pi h)\frac{dh}{dt}$

$= 9\pi \frac{dh}{dt}$

equating $9\pi\frac{dh}{dt} = 0.48\pi - 0.6\pi h$

$\therefore 9\frac{dh}{dt} = 0.48 - 0.6h$

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Question 8 continued

$\therefore \quad 9\frac{dh}{dt} = \frac{48}{100} - \frac{3h}{5}$

$\therefore \quad 900\frac{dh}{dt} = 48 - 60h$

$\therefore \quad 75\frac{dh}{dt} = 4 - 5h$

b) $75\int \frac{dh}{4-5h} = \int dt$

$\therefore -15\ln(4-5h) = t + c$

$t=0, h=0.2 \Rightarrow -15\ln 3 = c$

$\therefore \quad -15\ln(4-5h) = t - 15\ln 3$

when $h = 0.5$

$-15\ln 1.5 = t - 15\ln 3$

$\therefore \quad t = 15\ln 3 - 15\ln 1.5$

$= 15\ln 2$