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1. $z = 5 - 3\text{i}, \quad w = 2 + 2\text{i}$

Express in the form $a + b\text{i}$, where a and b are real constants,

(a) z^2, **(2)**

(b) $\frac{z}{w}$. **(3)**

a) $z^2 = (5-3i)^2 = 25 + 9i^2 - 30i$

$= 25 - 9 - 30i$

$= 16 - 30i$

b) $\frac{z}{w} = \frac{5-3i}{2+2i} = \frac{(5-3i)(2-2i)}{(2+2i)(2-2i)}$

$= \frac{10 + 6i^2 - 6i - 10i}{4 - 4i^2}$

$= \frac{10 - 6 - 6i - 10i}{4+4} = \frac{4 - 16i}{8}$

$= \frac{1}{2} - 2i$

Q1

(Total 5 marks)

N 3 5 4 0 6 A 0 2 3 2

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2.

$$\mathbf{A} = \begin{pmatrix} 2 & 0 \\ 5 & 3 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} -3 & -1 \\ 5 & 2 \end{pmatrix}$$

(a) Find **AB**.

(3)

Given that

$$\mathbf{C} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

(b) describe fully the geometrical transformation represented by **C**,

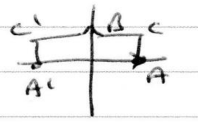
(2)

(c) write down $\mathbf{C}^{100}$.

(1)

(a) $AB = \begin{pmatrix} 2 & 0 \\ 5 & 3 \end{pmatrix}\begin{pmatrix} -3 & -1 \\ 5 & 2 \end{pmatrix} = \begin{pmatrix} -6 & -2 \\ 0 & 1 \end{pmatrix}$

(b) $\begin{pmatrix} 1 \\ 0 \end{pmatrix} \to \begin{pmatrix} -1 \\ 0 \end{pmatrix}$



$\begin{pmatrix} 0 \\ 1 \end{pmatrix} \to \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$\therefore$ C represents a reflection in the y-axis

(c) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Q2

(Total 6 marks)

3. $$f(x) = 5x^2 - 4x^{\frac{3}{2}} - 6, \quad x \geqslant 0$$

The root α of the equation $f(x) = 0$ lies in the interval $[1.6, 1.8]$.

(a) Use linear interpolation once on the interval $[1.6, 1.8]$ to find an approximation to α. Give your answer to 3 decimal places.
(4)

(b) Differentiate $f(x)$ to find $f'(x)$.
(2)

(c) Taking 1.7 as a first approximation to α, apply the Newton-Raphson process once to $f(x)$ to obtain a second approximation to α. Give your answer to 3 decimal places.
(4)

a) $f(1.6) = 12.8 - 8.0954308 - 6 = -1.2954308$
$f(1.8) = 16.2 - 9.6598137 - 6 = 0.5401863$

$$\alpha = 1.6 + \frac{1.2954308}{1.2954308 + 0.5401863} \times 0.2$$

$$= 1.6 + \frac{1.2954308}{1.8356171} \times 0.2$$

$= 1.6 + 0.7057195 \times 0.2$
$= 1.7411439$
$= 1.741 \quad (3 \text{ d.p.})$

b) $f'(x) = 10x - 6x^{1/2}$

c) $$x = 1.7 - \frac{f(1.7)}{f'(1.7)}$$

$$= 1.7 - \frac{14.45 - 8.8661153 - 6}{17 - 7.8230429}$$

$$= 1.7 - \frac{-0.4161152}{9.1769571} = 1.7 + 0.0453434$$

$= 1.7453435$
$= 1.745 \quad (3 \text{ d.p.})$

4. Given that $2 - 4\mathrm{i}$ is a root of the equation

$$z^2 + pz + q = 0,$$

where p and q are real constants,

(a) write down the other root of the equation, **(1)**

(b) find the value of p and the value of q. **(3)**

(a) $2 - 4i$

(b) $(z - (2+4i))(z-(2-4i)) = 0$

$\therefore\ z^2 - (2+4i)z - (2-4i)z + (2+4i)(2-4i) = 0$

$\therefore\ z^2 - 4z + 4 - 16i^2 = 0$

$\therefore\ z^2 - 4z + 20 = 0$

$\therefore\ p = -4,\ q = 20$

5. (a) Use the results for $\sum_{r=1}^{n} r$, $\sum_{r=1}^{n} r^2$ and $\sum_{r=1}^{n} r^3$, to prove that

$$\sum_{r=1}^{n} r(r+1)(r+5) = \frac{1}{4}n(n+1)(n+2)(n+7)$$

for all positive integers n.

(5)

(b) Hence, or otherwise, find the value of

$$\sum_{r=20}^{50} r(r+1)(r+5)$$

(2)

a) $\sum_1^n r(r+1)(r+5) = \sum_1^n r(r^2+6r+5) = \sum_1 r^3+6r^2+5r$

$= \sum_1^n r^3 + 6\sum_1^n r^2 + 5\sum_1^n r$

$= \frac{1}{4}n^2(n+1)^2 + 6\frac{n}{6}(n+1)(2n+1) + 5\frac{n}{2}(n+1)$

$= \frac{1}{4}n(n+1)\left[n(n+1) + 4(2n+1) + 10\right]$

$= \frac{1}{4}n(n+1)\left[n^2+n+8n+4+10\right]$

$= \frac{1}{4}n(n+1)(n^2+9n+14)$

$= \frac{1}{4}n(n+1)(n+2)(n+7)$

b) $\sum_{20}^{50} r(n+1)(r+5) = \sum_1^{50} r(r+1)(r+5) - \sum_1^{19} r(r+1)(r+5)$

$= \frac{1}{4}\times 50\times 51\times 52\times 57 - \frac{1}{4}\times 19\times 20\times 21\times 26$

$= 1889550 - 51870$

$= 1837680$

6.

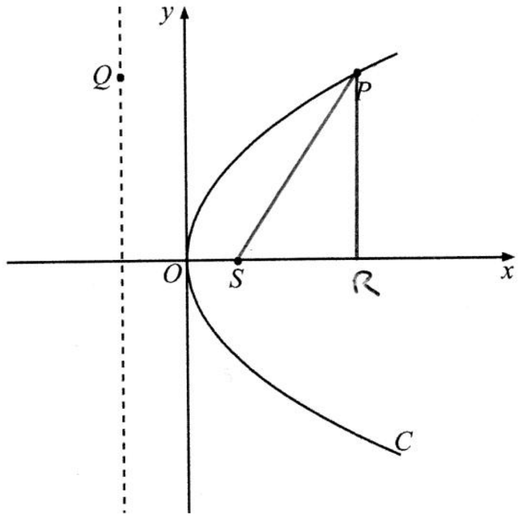


Figure 1

Figure 1 shows a sketch of the parabola C with equation $y^2 = 36x$.
The point S is the focus of C.

(a) Find the coordinates of S.
(1)

(b) Write down the equation of the directrix of C.
(1)

Figure 1 shows the point P which lies on C, where $y > 0$, and the point Q which lies on the directrix of C. The line segment QP is parallel to the x-axis.

Given that the distance PS is 25,

(c) write down the distance QP,
(1)

(d) find the coordinates of P,
(3)

(e) find the area of the trapezium $OSPQ$.
(2)

a) $y^2 = 36x$

$a = \frac{1}{4} \times 36 = 9$

$\therefore$ S is at $(9, 0)$

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Question 6 continued

b) $x = -9$

c) 25

d) ~~PQ~~ Let P have co-ordinates (x, y)

$PQ = x + 9$

but $PQ = 25$

$\therefore x = 16$

$PR^2 + SR^2 = PS^2$ (Pythagoras)

$\therefore y^2 + 7^2 = 25^2$

$\therefore y^2 = 24^2$

$\therefore y = \pm 24$ but $y > 0$ so $y = 24$

$\therefore$ P has co-ordinates $(16, 24)$

e) Area $= \frac{1}{2} \times 24 \, (25 + 9)$

$= 12 \times 34 = 408$

Q6

(Total 8 marks)

7.

$$z = -24 - 7\text{i}$$

(a) Show z on an Argand diagram.
(1)

(b) Calculate arg z, giving your answer in radians to 2 decimal places.
(2)

It is given that

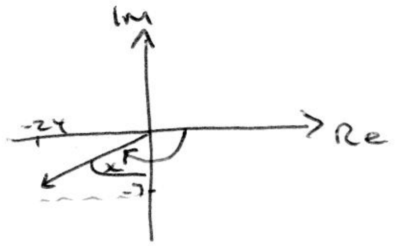
$$w = a + b\text{i}, \quad a \in \mathbb{R}, b \in \mathbb{R}$$

Given also that $|w| = 4$ and $\arg w = \dfrac{5\pi}{6}$,

(c) find the values of a and b,
(3)

(d) find the value of $|zw|$.
(3)

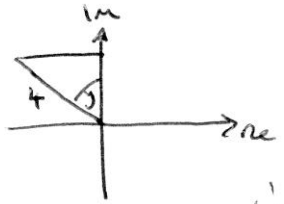
a)



b) $\tan x = \frac{24}{7} \Rightarrow x = 1.2870022$

$\therefore$ argument $= -\left(\frac{\pi}{2} + 1.2870022\right) = -2.86$ (2 d.p.)

c)



$y = \frac{5\pi}{6} - \frac{\pi}{2} = \frac{2\pi}{6} = \frac{\pi}{3}$

$\therefore w = -4\sin y + 4\cos y\, i$

$= -\frac{4\sqrt{3}}{2} + 4.0.2\, i$

$= -2\sqrt{3} + 2i$

$\therefore a = -2\sqrt{3}$

$b = 2$

Leave blank

Question 7 continued

d) $|zw| = |z||w|$

$|z| = \sqrt{7^2+24^2} = 25$

$\therefore |zw| = 25 \times 4 = 100$

8.

$$\mathbf{A} = \begin{pmatrix} 2 & -2 \\ -1 & 3 \end{pmatrix}$$

(a) Find det **A**. **(1)**

(b) Find $\mathbf{A}^{-1}$. **(2)**

The triangle R is transformed to the triangle S by the matrix **A**.
Given that the area of triangle S is 72 square units,

(c) find the area of triangle R. **(2)**

The triangle S has vertices at the points $(0,4)$, $(8,16)$ and $(12,4)$.

(d) Find the coordinates of the vertices of R. **(4)**

a) $6-2=4$

b) $\frac{1}{4}\begin{pmatrix} 3 & 2 \\ 1 & 2 \end{pmatrix}$

c) $\frac{72}{4} = 18$

d) $\frac{1}{4}\begin{pmatrix} 3 & 2 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 0 & 8 & 12 \\ 4 & 16 & 4 \end{pmatrix} = \begin{pmatrix} 2 & 14 & 11 \\ 2 & 10 & 5 \end{pmatrix}$

$\therefore$ co-ordinates of R are $(2,2)$ $(14,10)$ and $(11,5)$

9. A sequence of numbers $u_1, u_2, u_3, u_4, \ldots$ is defined by

$$u_{n+1} = 4u_n + 2, \quad u_1 = 2$$

Prove by induction that, for $n \in \mathbb{Z}^+$,

$$u_n = \frac{2}{3}\left(4^n - 1\right)$$

(5)

Let $n=1$ $\quad u_1 = \frac{2}{3}(4-1) = 2$ $\quad$ Given

Assume true for $n=r$ $\quad \therefore u_r = \frac{2}{3}(4^r - 1)$

$$\therefore u_{r+1} = 4\left(\frac{2}{3}(4^r-1)\right)+2$$

$$= \frac{8}{3}(4^r - 1) + 2$$

$$= \frac{2}{3}(4 \times 4^r - 4) + 2$$

$$= \frac{2}{3}(4^{r+1} - 4 + 3) \quad \text{since } \frac{2}{3} \cdot 3 = 2$$

$$= \frac{2}{3}(4^{r+1} - 1)$$

which is u_n with $r+1$ in place of n

This completes the proof by induction.

10. The point $P\left(6t, \dfrac{6}{t}\right), t \neq 0$, lies on the rectangular hyperbola H with equation $xy = 36$.

(a) Show that an equation for the tangent to H at P is

$$y = -\frac{1}{t^2}x + \frac{12}{t}$$

(5)

The tangent to H at the point A and the tangent to H at the point B meet at the point $(-9, 12)$.

(b) Find the coordinates of A and B.

(7)

a) $xy = 36$

$\Rightarrow y = \dfrac{36}{x}$

$\therefore \dfrac{dy}{dx} = -\dfrac{36}{x^2}$

At $(6t, \frac{6}{t})$ the gradient is $\dfrac{-36}{(6t)^2} = \dfrac{-1}{t^2}$

$\therefore$ the equation of the tangent is

$y = \dfrac{-1}{t^2}x + c$

But the line goes through $(6t, \frac{6}{t})$

$\therefore \dfrac{6}{t} = \dfrac{-1}{t^2} \times 6t + c$

$\therefore \dfrac{6}{t} = \dfrac{-6}{t} + c$

$\therefore c = \dfrac{12}{t}$

$\therefore$ equation is $y = -\dfrac{1}{t^2}x + \dfrac{12}{t}$

Leave blank

Question 10 continued

b) $12 = -\frac{1}{t^2} \times 9 + \frac{12}{t}$

$\therefore 12t^2 = 9 + 12t$

$\therefore 12t^2 - 12t - 9 = 0$

$\therefore 4t^2 - 4t - 3 = 0$

$\therefore (2t+1)(2t-3) = 0$

$\therefore t = -\frac{1}{2}$ or $t = \frac{3}{2}$

$\therefore$ co-ordinates are $(-3, -12)$ and $(9, 4)$